33. LINEAR PROGRAMMING

Linear Inequations in Two Variables

The inequalities of the form

$$ax + by \leq c,\ ax + by < c,\ ax + by \geq c \text{ and } ax + by > c$$

are called *linear inequations* in two variables x and y.

The points (x, y) for which the inequation is true, constitute its solution set.

Graph of a Linear Inequation

Let us consider an inequation, $ax + by \leq c$. For drawing its graph to obtain a solution set, we proceed as under.

STEP 1 Consider the equation $ax + by = c$, and plot the resulting line. In case of strict inequalities $<$ or $>$, draw the line as dotted, otherwise mark it thick.

STEP 2 Choose a point [if possible $(0, 0)$], not lying on this line. Substitute its coordinates in the inequation. If the inequation is satisfied then shade the portion of the plane which contains the chosen point; otherwise shade the portion which does not contain this point.

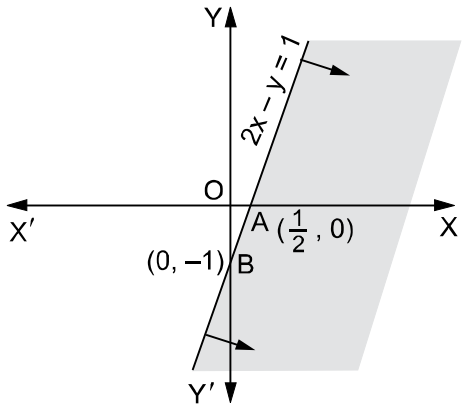
The shaded portion represents the solution set. The dotted line is not a part of the shaded region while the thick line is a part of it.

EXAMPLE *Graph the solution set of the inequation* $2x - y \geq 1$.

SOLUTION Consider the equation $2x - y = 1$.

We may write it as $\dfrac{x}{(1/2)} + \dfrac{y}{-1} = 1$.

This shows that the line $2x - y = 1$ makes intercepts of $\dfrac{1}{2}$ and -1 on the axes. Thus, the line meets the x-axis at $A\left(\dfrac{1}{2}, 0\right)$ and the y-axis at $B(0, -1)$. We plot these points and join them by a thick line.



Consider $O(0, 0)$. Clearly, $(0, 0)$ does not satisfy the given inequation.

So, out of the portions divided by this line, the one not containing $O(0, 0)$, together with the points on the line, forms the solution set.

Simultaneous Inequations

The solution set of a system of linear inequations in two variables is the set of all points (x, y) which satisfy all the inequations in the system simultaneously.

So, we find the region of the plane common to all the portions comprising the solution sets of the given inequations. When there is no region common to all the solutions of the given inequations, we say that the solution set of the system is empty.

The linear inequations are also known as *linear constraints.*

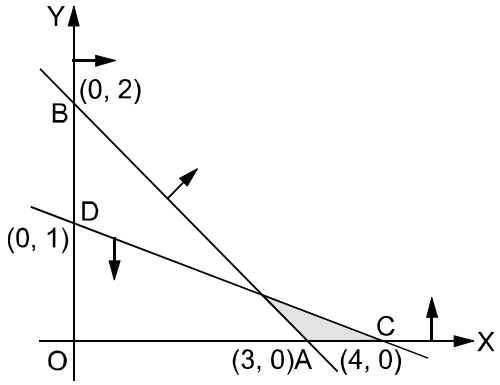
SOLVED EXAMPLES

EXAMPLE 1 *Draw the graph of the solution set of the system of inequations* $2x + 3y \geq 6,\ x + 4y \leq 4,\ x \geq 0$ *and* $y \geq 0$.

SOLUTION Consider the equations

$$2x + 3y = 6,\ x + 4y = 4,\ x = 0 \text{ and } y = 0.$$

Now, $2x + 3y = 6 \Rightarrow \frac{x}{3} + \frac{y}{2} = 1.$



This line meets the axes at $A(3, 0)$ and $B(0, 2)$. Join these points and draw a thick line. Clearly, the portion not containing $(0, 0)$ represents the solution set of the inequation $2x + 3y \geq 6$.

Again, $x + 4y = 4 \Rightarrow \frac{x}{4} + \frac{y}{1} = 1.$

This line meets the axes at $C(4, 0)$ and $D(0, 1)$. Join these points and draw a thick line. Clearly, the portion containing $(0, 0)$ represents the solution set of the inequation $x + 4y \leq 4$.

Clearly, $x \geq 0$ is represented by the y-axis and the portion on its right-hand side.

Also, $y \geq 0$ is represented by the x-axis and the portion above the x-axis.

Hence, the shaded region represents the solution set of the given inequations.

EXAMPLE 2 *Exhibit graphically the solution set of the system of linear inequations*

$$x + y \geq 1, 7x + 9y \leq 63, y \leq 5, x \leq 6, x \geq 0 \text{ and } y \geq 0.$$

SOLUTION $x + y = 1$ meets the axes at $A(1, 0)$ and $B(0, 1)$.

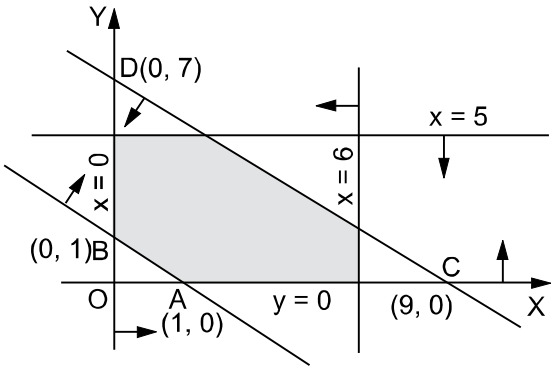
Join these points by a thick line. Clearly, the portion not containing $O(0, 0)$ is the solution set of $x + y \geq 1$.

$$7x + 9y = 63 \Rightarrow \frac{x}{9} + \frac{y}{7} = 1.$$

This line meets the axes at $C(9, 0)$ and $D(0, 7)$. Join these points by a thick line. Clearly, the portion containing $(0, 0)$ is the solution set of $7x + 9y \leq 63$.

$y = 5$ is a line parallel to the x-axis at a distance 5 from the x-axis and the portion containing $O(0, 0)$ is the solution set of the inequation $y \leq 5$.

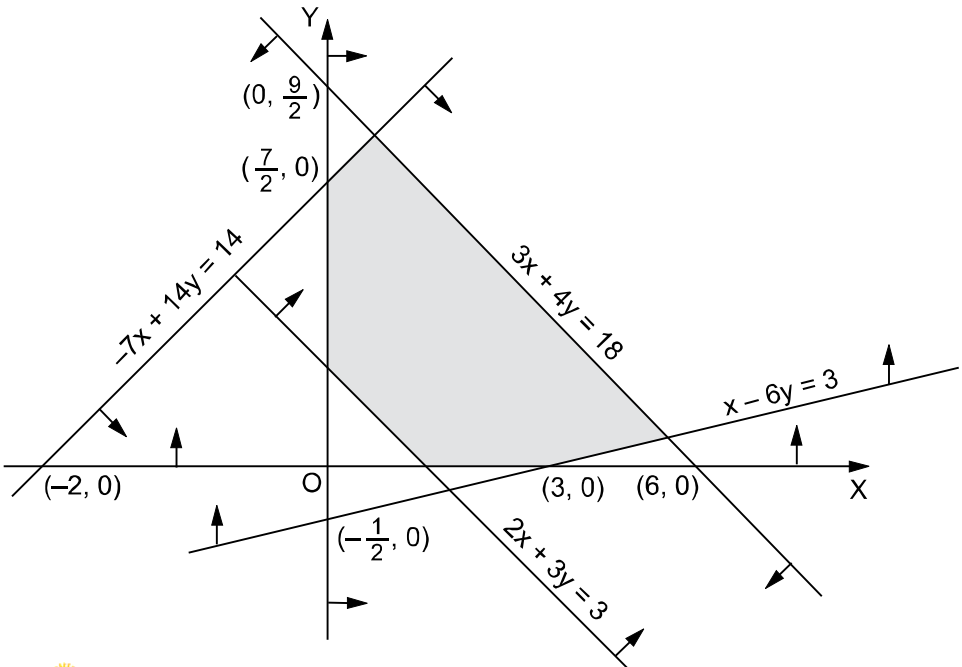
$x = 6$ is a line parallel to the y-axis at a distance 6 from the y-axis and the portion containing $(0, 0)$ is the solution set of $x \leq 6$.



Clearly, $x \geq 0$ has a solution represented by the y-axis and the portion on its right. Also, $y \geq 0$ has a solution represented by the x-axis and the portion above it.

The shaded region represents the solution set of the given system of inequations.

EXAMPLE 3 *Find the linear constraints for which the shaded area in the figure below is the solution set.*



SOLUTION Consider the line $3x + 4y = 18$.

Clearly, $O(0, 0)$ satisfies $3x + 4y \leq 18$.

Clearly, the shaded area and $(0, 0)$ lie on the same side of the line $3x + 4y = 18$.

So, we must have $3x + 4y \leq 18$.

Consider the line $x - 6y = 3$.

Clearly, $(0, 0)$ satisfies the inequation $x - 6y \leq 3$.

Also, the shaded area and $(0, 0)$ lie on the same side of the line $x - 6y = 3$.

So, we must have $x - 6y \leq 3$.

Consider the line $2x + 3y = 3$.

Clearly, $(0, 0)$ satisfies the inequation $2x + 3y \leq 3$.

But, the shaded region and the point $(0, 0)$ lie on the opposite sides of the line $2x + 3y = 3$.

So, we must have $2x + 3y \geq 3$.

Consider the line $-7x + 14y = 14$.

Clearly, $(0, 0)$ satisfies the inequation $-7x + 14y \leq 14$.

Also, the shaded region and the point $(0, 0)$ lie on the same side of the line $-7x + 14y = 14$.

So, we must have $-7x + 14y \leq 14$.

The shaded region is above the x-axis and on the right-hand side of the y-axis, so we have $y \geq 0$ and $x \geq 0$.

Thus, the linear constraints for which the shaded area in the given figure is the solution set, are

$3x + 4y \leq 18$, $x - 6y \leq 3$, $2x + 3y \geq 3$,
$-7x + 14y \leq 14$, $x \geq 0$ and $y \geq 0$.

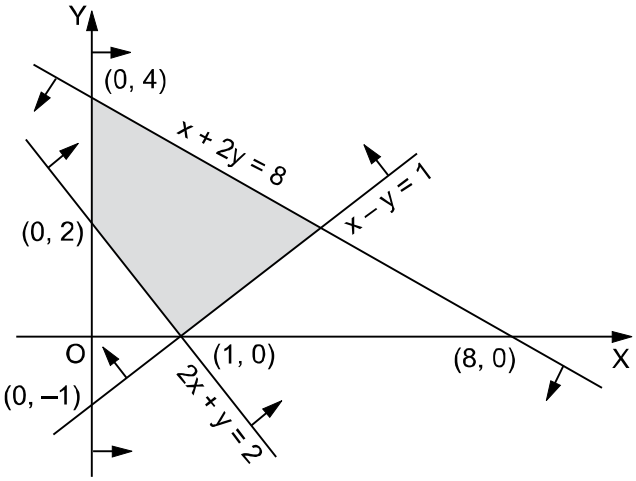
EXERCISE 33A

Graph the solution sets of the following inequations:

1. $x + y \geq 4$
2. $x - y \leq 3$
3. $x + 2y > 1$
4. $2x - 3y < 4$
5. $x \geq y - 2$
6. $y - 2 \leq 3x$

Solve each of the following systems of simultaneous inequations:

7. $2x + y > 1$ and $2x - y \geq -3$
8. $x - 2y \geq 0$, $2x - y \leq -2$
9. $3x + 4y \geq 12$, $x \geq 0$, $y \geq 1$ and $4x + 7y \leq 28$
10. Show that the solution set of the following linear constraints is empty:
    $x - 2y \geq 0$, $2x - y \leq -2$, $x \geq 0$ and $y \geq 0$.
11. Find the linear constraints for which the shaded area in the figure given is the solution set.



ANSWERS (EXERCISE 33A)

11. $x \geq 0, y \geq 0, 2x + y \geq 2, x - y \leq 1$ and $x + 2y \leq 8$

Linear Programming

Linear programming is the method used in decision making in business for obtaining the maximum or minimum value of a linear expression, subject to satisfying certain given linear inequations.

The linear expression is known as an *objective function* and the linear inequations are known as *linear constraints.*

LINEAR CONSTRAINTS In business or industry we want to make the best use of our limited resources like money, labour, time, materials, etc.

The limitations on the resources can often be expressed in the form of linear inequations, known as *linear constraints.*

OBJECTIVE FUNCTION *A linear function of the involved variables, which we want to maximize or minimize, subject to the given linear constraints, is known as an objective function.*

OPTIMAL VALUE OF AN OBJECTIVE FUNCTION *The maximum or minimum value of an objective function is known as its optimal value.*

FEASIBLE SOLUTION *A set of values of the variables satisfying all the constraints is known as a feasible solution of the system of inequations.*

OPTIMAL SOLUTION *A feasible solution which leads to the optimal value of an objective function is known as an optimal solution of the system of inequations.*

OPTIMIZATION TECHNIQUES *The processes of obtaining the optimal values of a system of inequations are called optimization techniques.*

A Linear Programming Problem (LPP)

A general linear programming problem consists of maximizing or minimizing an objective function, subject to certain given constraints.

Formulation of a Linear Programming Problem (LPP)

Working rules

STEP 1 Identify the unknowns in the given LPP. Denote them by x and y.

STEP 2 Formulate the objective function in terms of x and y. Be sure whether it is to be maximized or minimized.

STEP 3 Translate all the constraints in the form of linear inequations.

STEP 4 Solve these inequations simultaneously. Mark the common area by a shaded region. This is the feasible region.

STEP 5 Find the coordinates of all the vertices of the feasible region.

STEP 6 Find the value of the objective function at each vertex of the feasible region.

STEP 7 Find the values of x and y for which the objective function $Z = ax + by$ has maximum or minimum value (as the case may be).

Graphical Solution of an LPP

We shall restrict ourselves to the case of an LPP in two variables. We shall consider at least three constraints or inequations. Each inequation gives rise to a line in the plane. For a simultaneous solution of these inequations, we consider the region common to their solution sets. In each case we obtain such a region, a convex polygon, i.e., a closed region bounded by straight lines with the property that the line joining any two points of the region lies wholly in the region.

The maximum or minimum value of a linear function over a convex polygon occurs at some vertex of the polygon.

So, we look at the values of the objective function at the vertices of the set of feasible solutions. The largest of these values is the maximum value of the objective function and the smallest of these values is the minimum.

Graphical Method

It will be clear from the following solved examples.

SOLVED EXAMPLES

EXAMPLE 1 *Solve the following problem graphically:*
Minimize and maximize $z = 3x + 9y$, *subject to the constraints*
$x + 3y \le 60,\ x + y \ge 10,\ x \le y,\ x \ge 0$ *and* $y \ge 0$.

SOLUTION ***Region represented by*** $x + 3y \le 60$
Consider the equation $x + 3y = 60$.

$$x = 0 \Rightarrow 3y = 60 \Rightarrow y = 20; y = 0 \Rightarrow x = 60$$

Plot the points $A(0, 20)$ and $B(60, 0)$. Join AB and produce it both ways.

Putting $x = 0$ and $y = 0$, we get $0 + 3 \times 0 = 0 \leq 60$.

$\therefore$ $O(0, 0)$ lies in the region $x + 3y \leq 60$.

So, the region containing the origin is the solution set of $x + 3y \leq 60$.

Region represented by $x + y \geq 10$

Consider the equation $x + y = 10$.

$x = 0 \Rightarrow y = 10; y = 0 \Rightarrow x = 10$.

Plot the points $C(0, 10)$ and $D(10, 0)$. Join CD and produce it both ways.

Now, $x = 0, y = 0 \Rightarrow 0 + 0 \geq 10$ is not true.

$\therefore$ $O(0, 0)$ does not lie in the region $x + y \geq 10$.

Region represented by $x \leq y$, i.e., $x - y \leq 0$

Consider the equation $x = y$, i.e., $x - y = 0$.

Clearly, $x = 10 \Rightarrow y = 10$. And, $x = 20 \Rightarrow y = 20$.

Plot the points $E(10, 10)$ and $F(20, 20)$. Join EF and produce it both ways.

Clearly, $O(0, 0)$ satisfies $x - y \leq 0$.

$\therefore$ $O(0, 0)$ lies in the region $x - y \leq 0$.

We know that:

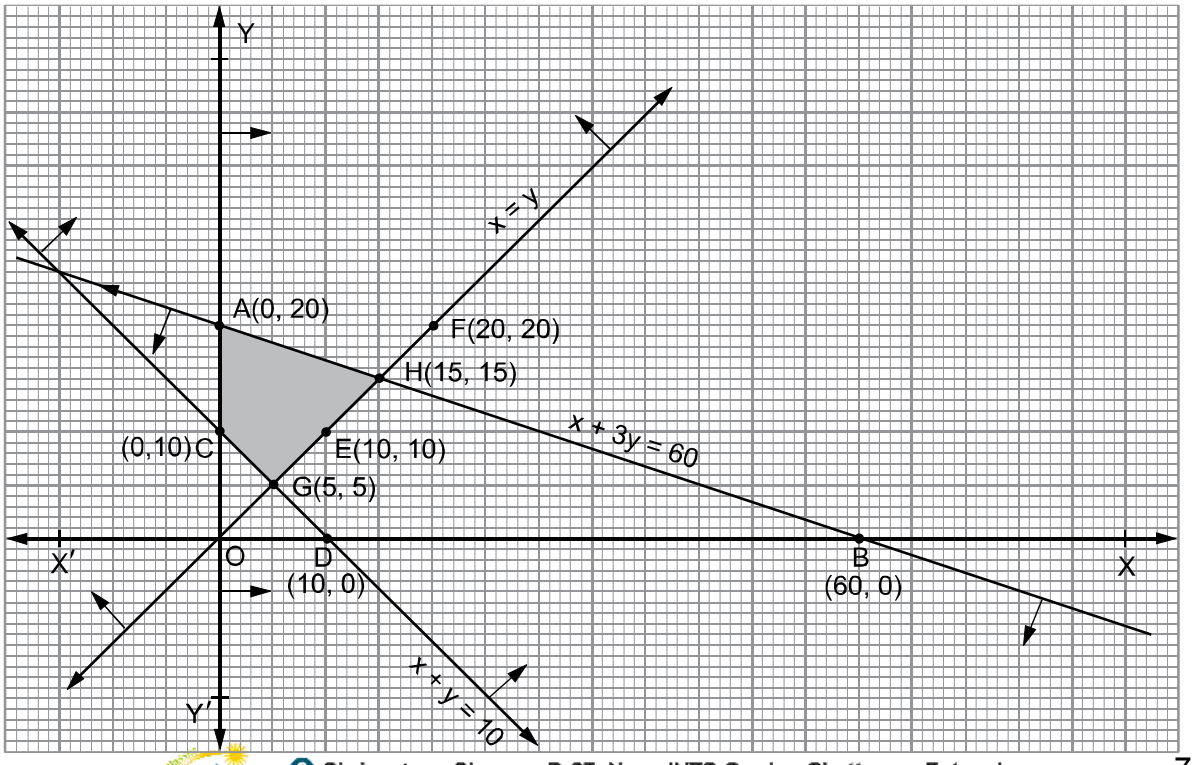
$x \geq 0$ is the y-axis and the region on its RHS.

$y \geq 0$ is the x-axis and the region above the x-axis.

On solving $x = y$ and $x + y = 10$, we get the point $G(5, 5)$.

On solving $x = y$ and $x + 3y = 60$, we get $H(15, 15)$.

Thus, the feasible region is $ACGH$, as shown in the figure.



Value of $z = 3x + 9y$:

(i) At $A(0, 20)$ it is $(3 \times 0 + 9 \times 20) = 180$.

(ii) At $C(0, 10)$ it is $(3 \times 0 + 9 \times 10) = 90$.

(iii) At $G(5, 5)$ it is $(3 \times 5 + 9 \times 5) = 60$.

(iv) At $H(15, 15)$ it is $(3 \times 15 + 9 \times 15) = 180$.

So, the minimum value of z is 60 and its maximum value is 180.

EXAMPLE 2 *A furniture dealer deals in only two items: tables and chairs. He has* ₹ 5000 *to invest and a space to store at most* 60 *pieces. A table costs him* ₹ 250 *and a chair,* ₹ 50. *He can sell a table at a profit of* ₹ 50 *and a chair at a profit of* ₹ 15. *Assuming that he can sell all the items that he buys, how should he invest his money in order that he may maximize his profit?* **[CBSE 2006C]**

SOLUTION This problem can be formulated as under.

Let x and y be the required numbers of tables and chairs respectively. Then, clearly we have

$x \geq 0, y \geq 0; x + y \leq 60;$

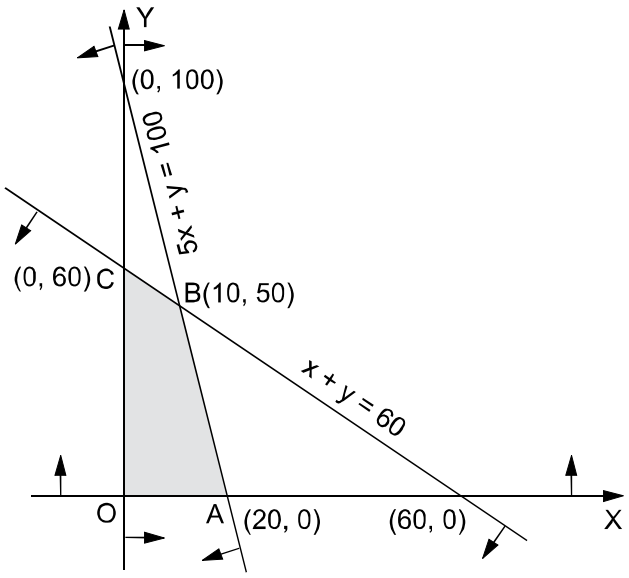
$250x + 50y \leq 5000$, i.e., $5x + y \leq 100$.

Let P be the profit function. Then, $P = 50x + 15y$.

Now, we have to maximize P.

Now, $x + y = 60 \Rightarrow \frac{x}{60} + \frac{y}{60} = 1$.

This line meets the axes at (60, 0) and (0, 60). Plot these points and join them to get the line $x + y = 60$.



Also, $5x + y = 100 \Rightarrow \frac{x}{20} + \frac{y}{100} = 1$.

This line meets the axes at (20, 0) and (0, 100). Plot these points an

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New Delhi - 110074
9810934436 , 8076575278 , 8700391727

join them to get the line $5x + y = 100$.

Also, the line $x = 0$ is the y-axis and the line $y = 0$ is the x-axis.

These four straight lines enclose the quadrilateral $OABC$.

The coordinates of the points O, A, B, C are (0, 0), (20, 0), (10, 50) and (0, 60) respectively.

At these points, the corresponding values of $P = 50x + 15y$ are 0, 1000, 1250 and 900 respectively.

Clearly, it is maximum at $B(10, 50)$.

So, for a maximum profit, the dealer should purchase 10 tables and 50 chairs.

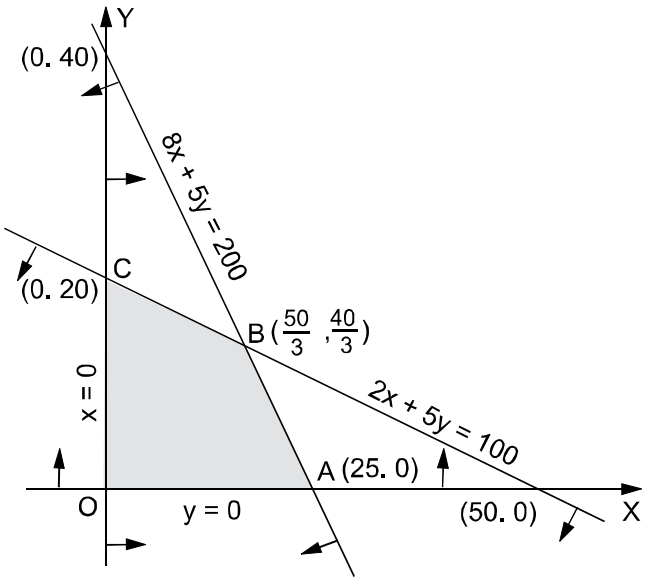
EXAMPLE 3 *If a young man rides his motorcycle at* 25 km *per hour, he has to spend* ₹ 2 *per kilometre on petrol; if he rides it at a faster speed of* 40 km *per hour, the petrol cost increases to* ₹ 5 *per kilometre. He has* ₹ 100 *to spend on petrol and wishes to find the maximum distance he can travel within one hour. Express this as a linear programming problem and then solve it.* **[CBSE 2013C]**

SOLUTION Suppose that the young man rides x km at 25 km per hour and y km at 40 km per hour. Then, we have to maximize $P = x + y$.

Clearly, $x \geq 0$, $y \geq 0$, $2x + 5y \leq 100$.

Since the available time is at most one hour, we have

$$\frac{x}{25} + \frac{y}{40} \leq 1 \text{ or } 8x + 5y \leq 200.$$



Now, we solve the system of the inequations.

$$2x + 5y = 100 \Rightarrow \frac{x}{50} + \frac{y}{20} = 1.$$

This line meets the axes at (50, 0) and (0, 20). Plot these points and join them to get the line $2x + 5y = 100$.

Also, $8x + 5y = 200 \Rightarrow \frac{x}{25} + \frac{y}{40} = 1.$

This line meets the axes at (25, 0) and (0, 40). Plot these points and obtain the line $8x + 5y = 200$.

$x = 0$ is the y-axis and $y = 0$ is the x-axis.

We find that the solution set of the above system is the shaded region $OABC$.

The coordinates of O, A, B, C are (0, 0), (25, 0), $\left(\frac{50}{3}, \frac{40}{3}\right)$ and (0, 20) respectively.

The values of $P = x + y$ at these points are 0, 25, 30 and 20 respectively.

So, $P = x + y$ is maximum when $x = \frac{50}{3}$ and $y = \frac{40}{3}$.

Thus, the young man can cover the maximum distance of 30 km, if he rides $\frac{50}{3}$ km at 25 km/h and $\frac{40}{3}$ km at 40 km/h.

EXAMPLE 4 *Suppose every gram of wheat provides* 0.1 g *of proteins and* 0.25 g *of carbohydrates, and the corresponding values for rice are* 0.05 g *and* 0.5 g *respectively. Wheat costs* ₹ 5 *and rice* ₹ 20 *per kilogram. The minimum daily requirements of proteins and carbohydrates for an average man are* 50 g *and* 200 g *respectively. In what quantities should wheat and rice be mixed in the daily diet to provide the minimum daily requirements of proteins and carbohydrates at minimum cost, assuming that both wheat and rice are to be taken in the diet?*

SOLUTION Let x g of wheat and y g of rice be mixed to fulfil the requirements.

Then, we have to minimize the cost function

$$Z = \frac{5x}{1000} + \frac{20y}{1000}, \text{ i.e., } Z = \frac{x}{200} + \frac{y}{50}. \quad \ldots \text{(i)}$$

x g of wheat and y g of rice must give at least 50 g of proteins.

So, we must have

$$0.1x + 0.05y \geq 50 \text{ or } 2x + y \geq 1000.$$

Similarly, x g of wheat and y g of rice must give at least 200 g of carbohydrates.

So, we must have

$$0.25x + 0.5y \geq 200 \text{ or } x + 2y \geq 800.$$

Thus, we have to minimize $Z = \frac{x}{200} + \frac{y}{50}$, subject to the constraints

$$x > 0, y > 0, 2x + y \geq 1000 \text{ and } x + 2y \geq 800.$$

$2x + y = 1000 \Rightarrow \frac{x}{500} + \frac{y}{1000} = 1.$

This line meets the axes at $A(500, 0)$ and $B(0, 1000)$.

Plot these points and join them to obtain the line $2x + y = 1000$.

Clearly, $(0, 0)$ does not satisfy $2x + y \geq 1000$.

Again, $x + 2y = 800 \Rightarrow \frac{x}{800} + \frac{y}{400} = 1$.

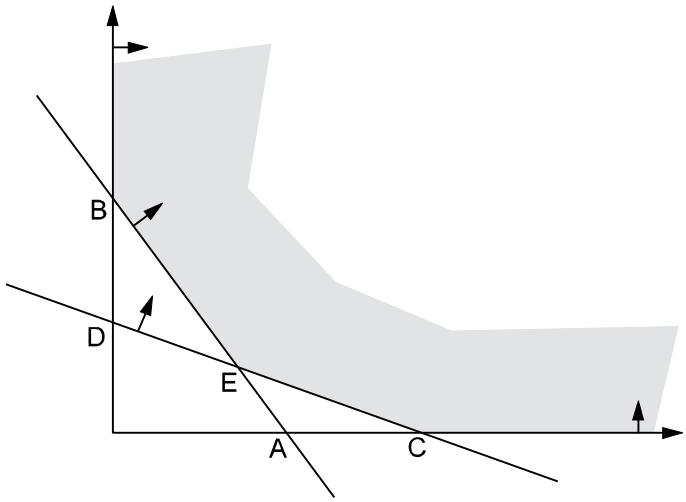
This line meets the axes at $C(800, 0)$ and $D(0, 400)$.

Plot these points and join them to obtain the line $x + 2y = 800$.

Clearly, $(0, 0)$ does not satisfy $x + 2y \geq 800$.

$x = 0$ is the y-axis and $y = 0$ is the x-axis.

We obtain the solution set of the above system, as shown by the shaded region.



Solving $2x + y = 1000$ and $x + 2y = 800$, we get the point of intersection of AB and CD, given by $E(400, 200)$.

The minimum value of $Z = \frac{x}{200} + \frac{y}{50}$ would be at some vertex of the unbounded feasible region BEC.

Clearly, at B we have $x = 0$, and at C we have $y = 0$.

Also, the value of Z at $E(400, 200) = \frac{400}{200} + \frac{200}{50} = 6$.

So, we must have 400 g of wheat and 200 g of rice.

EXAMPLE 5 *A firm manufactures two types of products, A and B, and sells them at a profit of ₹ 5 per unit of type A and ₹ 3 per unit of type B. Each product is processed on two machines, M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2; whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product should the firm produce a day in order to maximize the profit. Solve the problem graphically.* **[CBSE 2000]**

SOLUTION Let x units of A and y units of B be produced in order to have a maximum profit.

Then, clearly $x \geq 0$ and $y \geq 0$.

x units of A and y units of B will take $(x + y)$ minutes on M_1.

$\therefore \quad x + y \leq 300$.

x units of A and y units of B will take $(2x + y)$ minutes on M_2.

$\therefore \quad 2x + y \leq 360$.

Let Z be the profit function. Then, $Z = 5x + 3y$.

We have to maximize $Z = 5x + 3y$, subject to the constraints

$x \geq 0, \quad y \geq 0, \quad x + y \leq 300$ and $2x + y \leq 360$.

Now, $x + y = 300 \quad \Rightarrow \quad \frac{x}{300} + \frac{y}{300} = 1$.

This line meets the axes in (300, 0) and (0, 300).

Joining these points, we get the line $x + y = 300$.

Since (0, 0) satisfies the inequation $x + y \leq 300$, the region below the line $x + y = 300$ containing $O(0, 0)$ represents $x + y \leq 300$.

Again, $2x + y = 360 \Rightarrow \frac{x}{180} + \frac{y}{360} = 1$.

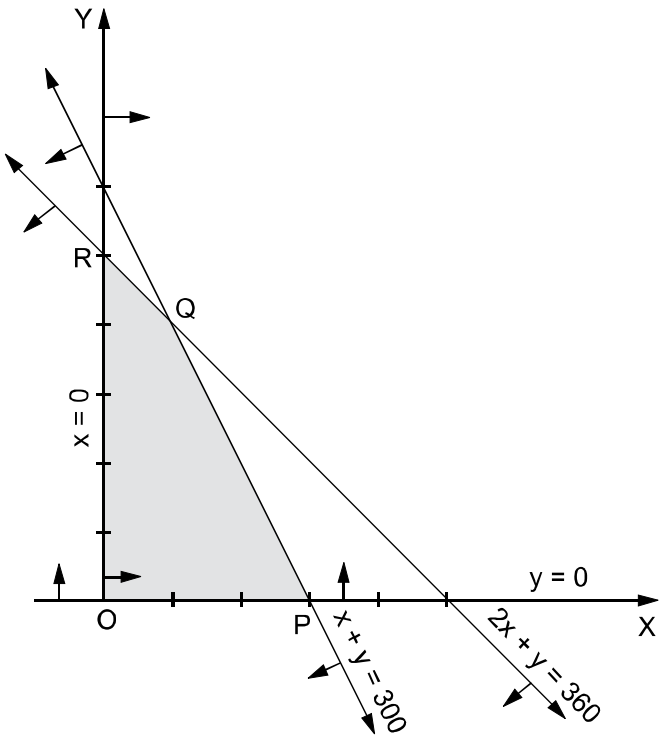
This line meets the axes in (180, 0) and (0, 360).

Joining these points, we get the line $2x + y = 360$.

Since (0, 0) satisfies $2x + y \leq 360$, the region below the line $2x + y = 360$ containing (0, 0) represents $2x + y \leq 360$.

Also $x = 0$ is the y-axis and $y = 0$ is the x-axis.

On drawing these lines and shading the feasible region, we obtain a figure, given below.



On solving $x = 0$ and $x + y = 300$, we get the point $R(0, 300)$.

On solving $x + y = 300$ and $2x + y = 360$, we get the point $Q(60, 240)$.

$\therefore$ the vertices of the feasible region are

$O(0, 0)$, $P(180, 0)$, $Q(60, 240)$ and $R(0, 300)$.

Values of $Z = 5x + 3y$ at O, P, Q, R are 0, 900, 1020, 900 respectively.

$\therefore$ Z is maximum when $x = 60$ and $y = 240$.

EXAMPLE 6 *An aeroplane of an airline can carry a maximum of* 200 *passengers. A profit of* ₹ 400 *is made on each first-class ticket and a profit of* ₹ 300 *is made on each economy-class ticket. The airline reserves at least* 20 *seats for first class. However, at least* 4 *times as many passengers prefer to travel by economy class than by first class. Determine how many of each type of tickets must be sold in order to maximize the profit for the airline. What is the maximum profit?*

SOLUTION Let x tickets of first class and y tickets of economy class be sold to maximize the profit. Then,

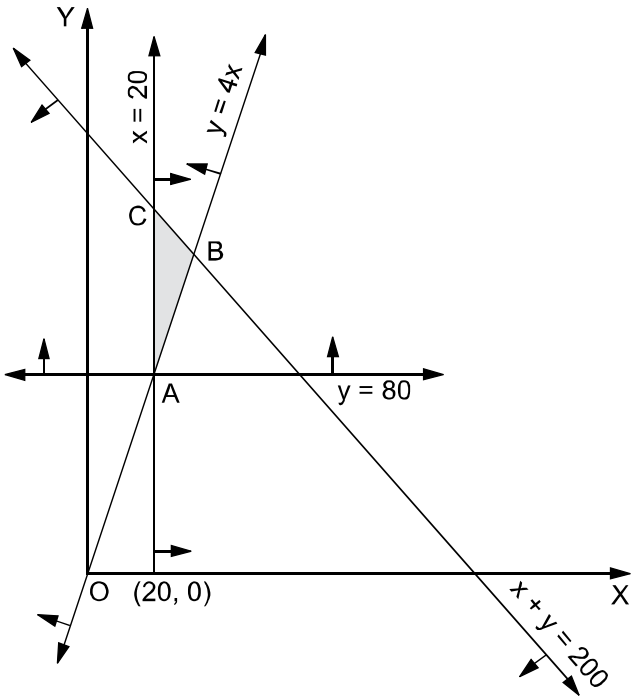
$$x \geq 20,\ y \geq 4x,\ y \geq 80 \text{ and } x + y \leq 200.$$

The profit function is given by $Z = 400x + 300y$.

Draw the graphs of the lines

$$x = 20,\ y = 4x,\ y = 80 \text{ and } x + y = 200$$

as shown below.



Graph of the inequation $x \geq 20$

Since (0, 0) does not satisfy $x \geq 20$, the line $x = 20$ together with the region to its right-hand side, not containing (0, 0), represents the region $x \geq 20$.

Graph of the inequation $y \geq 4x$

Clearly, since (20, 0) does not satisfy the inequation $y \geq 4x$, the line

$y = 4x$ together with the region to its left, not containing $(20, 0)$, represents $y \geq 4x$.

Graph of the inequation $y \geq 80$

Clearly, the line $y = 80$ and the region above this line represents $y \geq 80$.

Graph of the inequation $x + y \leq 200$

Clearly, $(0, 0)$ satisfies $x + y \leq 200$. So, the line $x + y = 200$ together with the region containing $O(0, 0)$ represents $x + y \leq 200$.

Thus, the shaded region in the given figure is the feasible region, whose vertices are A, B and C. A is the point of intersection of $x = 20$ and $y = 80$.

So, its coordinates are $A(20, 80)$.

On solving $y = 4x$ and $x + y = 200$, we get $B(40, 160)$.

On solving $x = 20$ and $x + y = 200$, we get $C(20, 180)$.

The values of $Z = 400x + 300y$ at $A(20, 80)$, $B(40, 160)$ and $C(20, 180)$ are respectively ₹ 32000, ₹ 64000 and ₹ 62000.

$\therefore$ Z is maximum at $x = 40$, $y = 160$.

EXAMPLE 7 *A chemical industry produces two compounds, A and B. The following table gives the units of ingredients C and D (per* kg*) of compounds A and B as well as minimum requirements of C and D, and costs per* kg *of A and B.*

	Compound (in units)		*Minimum requirement (in units)*
	A	*B*	
Ingredient C (per kg*)*	1	2	80
Ingredient D (per kg*)*	3	1	75
Cost per kg (in ₹)	4	6	

Find the quantities of A and B which would minimize the cost.

SOLUTION Let x kg of A and y kg of B be produced. Then,

$$x \geq 0,\ y \geq 0,\ x + 2y \geq 80 \text{ and } 3x + y \geq 75.$$

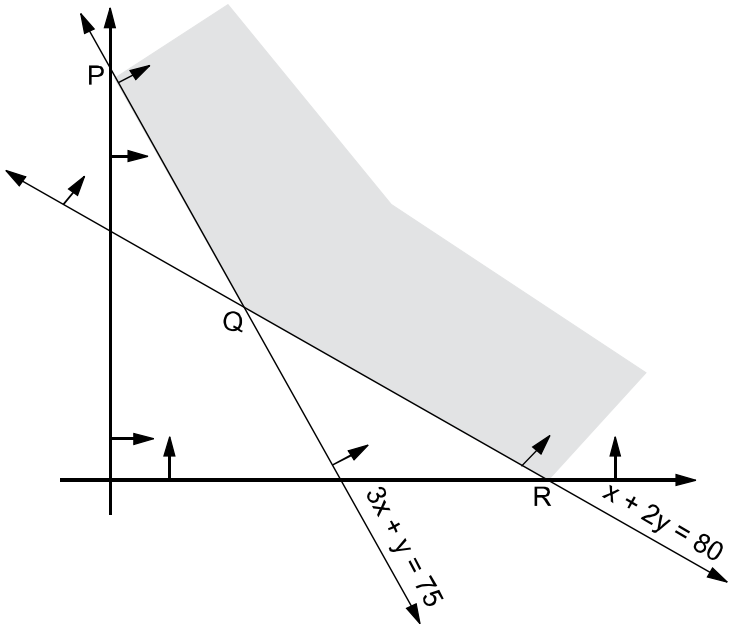
Then cost function is given by $Z = 4x + 6y$.

Thus, we have to minimize $Z = 4x + 6y$, subject to the constraints:

$$x \geq 0,\ y \geq 0,\ x + 2y \geq 80 \text{ and } 3x + y \geq 75.$$

Draw the graphs of the lines

$$x = 0,\ y = 0,\ x + 2y = 80 \text{ and } 3x + y = 75.$$



Since (0, 0) does not satisfy the inequation $x + 2y \geq 80$, the line $x + 2y = 80$ together with the region not containing (0, 0) represents $x + 2y \geq 80$.

Since (0, 0) does not satisfy the inequation $3x + y \geq 75$, the line $3x + y = 75$ together with the region not containing (0, 0) represents $3x + y \geq 75$.

Thus, the shaded region is the feasible region.

The vertices of this region are P, Q and R.

On solving $x = 0$ and $3x + y = 75$, we get the point $P(0, 75)$.

On solving $x + 2y = 80$ and $3x + y = 75$, we get the point $Q(14, 33)$.

On solving $y = 0$ and $x + 2y = 80$, we get the point $R(80, 0)$.

The values of $Z = 4x + 6y$ at the points $P(0, 75)$, $Q(14, 33)$ and $R(80, 0)$ are 450, 254 and 320 respectively.

Thus, Z is minimum at $Q(14, 33)$.

Hence, for a minimum cost, 14 kg of A and 33 kg of B must be taken.

EXAMPLE 8 *A company makes two types of belts, A and B; profits on these belts being* ₹ 4 *and* ₹ 3 *each respectively. Each belt of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make* 1000 *belts per day. The supply of leather is sufficient for only* 800 *belts per day (both A and B combined). At the most* 400 *buckles for belts of type A and* 700 *for those of type B are available per day. How many belts of each type should the company make per day so as to maximize the profit?*

SOLUTION Let x belts of type A and y belts of type B be made.

Then, $x \geq 0$, $y \geq 0$, $x \leq 400$, $y \leq 700$ and $x + y \leq 800$.

Now, 1000 belts of type B can be made in 1 day.

$\therefore$ 500 belts of type A can be made in 1 day.

$\therefore$ time taken to make x belts of type A and y belts of type B

$$= \left(\frac{x}{500} + \frac{y}{1000}\right) \text{ days.}$$

$\therefore$ $\frac{x}{500} + \frac{y}{1000} \leq 1$, i.e., $2x + y \leq 1000$.

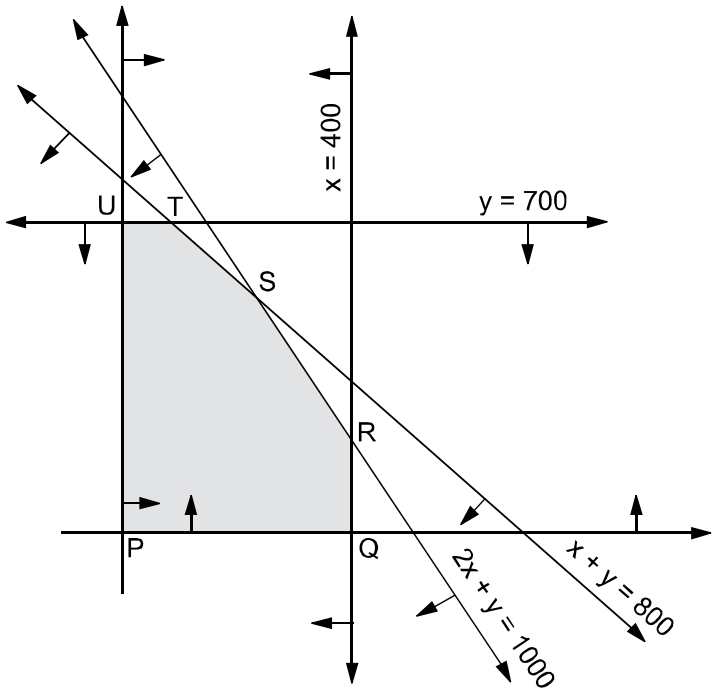
We have to maximize $Z = 4x + 3y$, subject to the constraints

$x \geq 0$, $y \geq 0$, $x \leq 400$, $y \leq 700$, $x + y \leq 800$ and $2x + y \leq 1000$.

We draw the graphs of the lines

$x = 0$, $y = 0$, $x = 400$, $y = 700$, $x + y = 800$, $2x + y = 1000$

as shown below.



Since (0, 0) satisfies $x \leq 400$, the line $x = 400$ together with the region containing $O(0, 0)$ represents $x \leq 400$.

Since (0, 0) satisfies $y \leq 700$, the line $y = 700$ together with the region containing $O(0, 0)$ represents $y \leq 700$.

Since (0, 0) satisfies $x + y \leq 800$, the line $x + y = 800$ together with the region containing $O(0, 0)$ represents $x + y \leq 800$.

Since (0, 0) satisfies $2x + y \leq 1000$, the line $2x + y = 1000$ together with the region containing $O(0, 0)$ represents $2x + y \leq 1000$.

The y-axis and the region to its right-hand side represents $x \geq 0$.

The x-axis and the region above it represents $y \geq 0$.

Thus, the shaded region represents the feasible region, whose vertices are P, Q, R, S, T and U.

Clearly, the coordinates of P and Q are (0, 0) and (400, 0) respectively.

On solving $x = 400$ and $2x + y = 1000$, we get $R(400, 200)$.

On solving $x + y = 800$ and $2x + y = 1000$, we get $S(200, 600)$.

On solving $y = 700$ and $x + y = 800$, we get $T(100, 700)$.

On solving $x = 0$ and $y = 700$, we get $U(0, 700)$.

The values of $Z = 4x + 3y$ at the points P, Q, R, S, T and E are respectively 0, 1600, 2200, 2600, 2500 and 2100.

The maximum of these values is 2600 occurring at $S(200, 600)$.

$\therefore$ Z is maximum when $x = 200$ and $y = 600$.

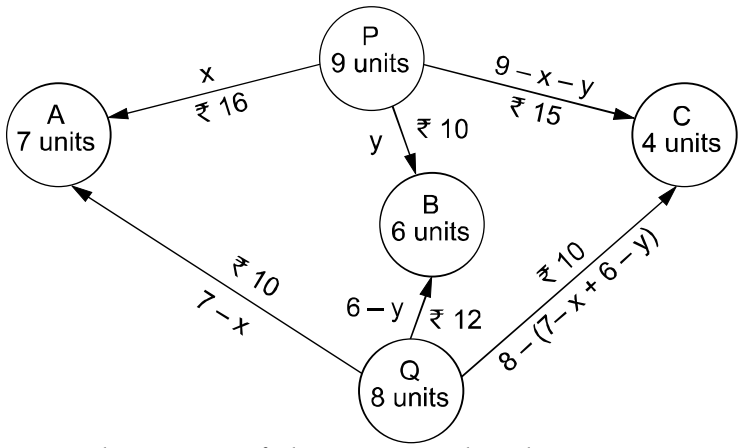
Thus, the company should make 200 belts of type A and 600 belts of type B to have a maximum profit.

EXAMPLE 9 *A company has factories located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 7, 6 and 4 units of the commodity while the weekly production capacities of the factories at P and Q are respectively 9 and 8 units. The cost of transportation per unit is given below.*

From \ *To*	*Cost (in ₹)*		
	A	*B*	*C*
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum? Formulate the above LPP mathematically and then solve it.

SOLUTION This problem can be explained diagramatically as follows:



Let x units and y units of the commodity be transported from the factory at P to the depots at A and B respectively. Then, $(9 - x - y)$ units will be transported from the factory at P to the depot at C.

$\therefore$ $x \geq 0, y \geq 0$, and $9 - x - y \geq 0 \Rightarrow x + y \leq 9$.

The weekly requirement of the depot at A is 7 units. So, $(7 - x)$ units will be transported to A from the factory at Q.

Similarly, $(6 - y)$ units will be transported to B from the factory at Q.

And, $8-(7-x+6-y)=(x+y-5)$ units will be transported to C from the factory at Q.

$\therefore \quad 7-x \geq 0,\ 6-y \geq 0$ and $x+y-5 \geq 0$,

i.e., $x \leq 7,\ y \leq 6$ and $x+y \geq 5$.

The total cost of transportation is

$$Z = 16x + 10(7-x) + 10y + 12(6-y) + 15(9-x-y) + 10(x+y-5)$$

$$\Rightarrow Z = x - 7y + 227.$$

Now, we have to find the values of x and y which minimize

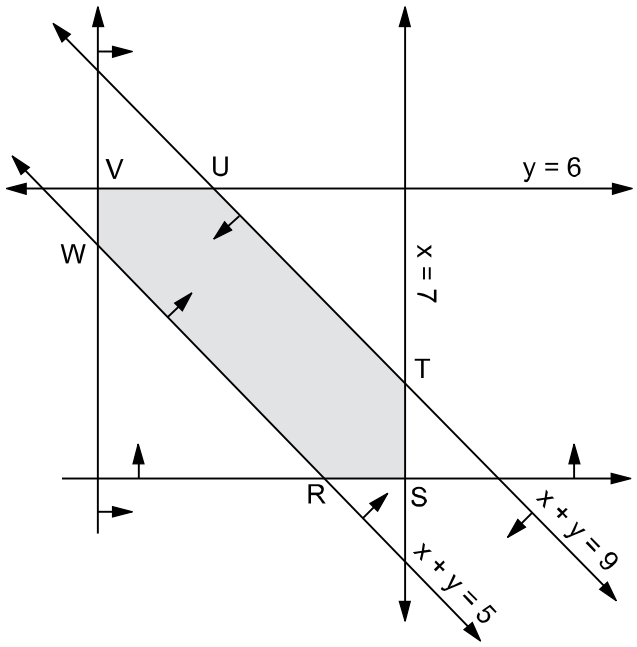
$Z = x - 7y + 227$, subject to the constraints

$x \geq 0,\ y \geq 0,\ x+y \leq 9,\ x \leq 7,\ y \leq 6$ and $x+y \geq 5$.

We draw the graphs of the lines

$x=0,\ y=0,\ x+y=9,\ x=7,\ y=6$ and $x+y=5$

as shown below.



Since (0, 0) satisfies $x + y \leq 9$, the line $x + y = 9$ together with the region containing $O(0,0)$ represents $x + y \leq 9$.

Since (0, 0) does not satisfy $x + y \geq 5$, the line $x + y = 5$ together with the region not containing $O(0,0)$ represents $x + y \geq 5$.

Since (0, 0) satisfies $x \leq 7$, the line $x = 7$ together with the region containing $O(0, 0)$ represents $x \leq 7$.

Since (0, 0) satisfies $y \leq 6$, the line $y = 6$ together with the region containing $O(0, 0)$ represents $y \leq 6$.

The y-axis and the region to its right-hand side represents $x \geq 0$.

The x-axis and the region above it represents $y \geq 0$.

Thus, the shaded region represents the feasible region whose vertices are R, S, T, U, V and W.

On solving $y = 0$ and $x + y = 5$, we get $R(5, 0)$.

On solving $y = 0$ and $x + 7 = 0$, we get $S(7, 0)$.

On solving $x = 7$ and $x + y = 9$, we get $T(7, 2)$.

On solving $x + y = 9$ and $y = 6$, we get $U(3, 6)$.

On solving $x = 0$ and $y = 6$, we get $V(0, 6)$.

On solving $x = 0$ and $x + y = 5$, we get $W(0, 5)$.

The values of $Z = x - 7y + 227$ at R, S, T, U, V and W are

232, 234, 220, 188, 185 and 192 respectively.

Thus, Z is minimum at $V(0, 6)$, i.e., when $x = 0$ and $y = 6$.

$\therefore$ from the factory at P there must be a delivery of 0, 6 and 3 units to A, B, C respectively.

And, the factory at Q must deliver 7, 0 and 1 units to A, B, C respectively.

EXAMPLE 10 *A factory owner purchases two types of machines A and B for his factory. The requirements and the limitations for the machines are as follows:*

Machine	*Area occupied*	*Labour force on each machine*	*Daily output (in units)*
A	1000 m^2	12 men	60
B	1200 m^2	8 men	40

He has maximum area of 9000 m^2 *available and* 72 *skilled labourers who can operate both the machines. How many machines of each type should he buy to maximize the daily output?* **[CBSE 2008]**

SOLUTION Let x machines of type A and y machines of type B be bought and let z be the daily output.

Then, $z = 60x + 40y$. … (i)

Maximum area available = 9000 m^2.

$\therefore \quad 1000x + 1200y \leq 9000$

$\Rightarrow \quad 5x + 6y \leq 45$. … (ii)

Maximum labour available = 72 men.

$\therefore \quad 12x + 8y \leq 72 \quad \Rightarrow \quad 3x + 2y \leq 18$. … (iii)

Now, we have to maximize $z = 60x + 40y$, subject to the constraints

$$5x + 6y \leq 45,$$
$$3x + 2y \leq 18,$$
$$x \geq 0 \text{ and } y \geq 0.$$

Now, $5x + 6y = 45 \quad \Rightarrow \quad \frac{x}{9} + \frac{y}{(15/2)} = 1.$

This line meets the axes at $A(9, 0)$ and $B\left(0, \frac{15}{2}\right)$.

Plot these points and join them to obtain the line $5x + 6y = 45$.

Clearly, (0, 0) satisfies $5x + 6y \le 45$.

So, the region below AB represents $5x + 6y \le 45$.

Again, $3x + 2y = 18 \Rightarrow \frac{x}{6} + \frac{y}{9} = 1$.

This line meets the axes at $C(6, 0)$ and $D(0, 9)$.

Plot these points and join them to obtain the line $3x + 2y = 18$.

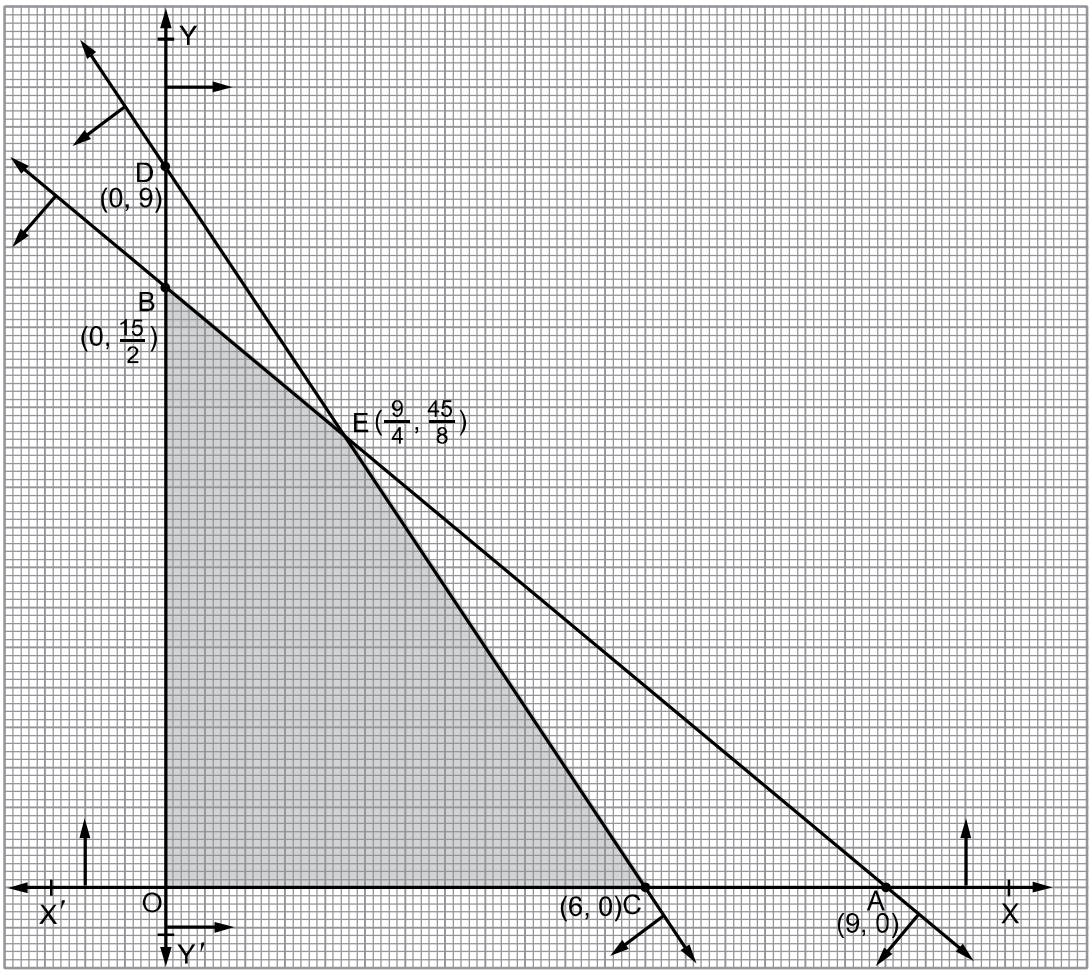
Clearly, (0, 0) satisfies $3x + 2y \le 18$.

So, the region below CD represents $3x + 2y \le 18$.

$x \ge 0$ is the region to the right of the y-axis.

And, $y \ge 0$ is the region above the x-axis.

On solving $5x + 6y = 45$ and $3x + 2y = 18$ simultaneously, we get $x = \frac{9}{4}$ and $y = \frac{45}{8}$.



So, the lines AB and CD intersect at $E\left(\frac{9}{4}, \frac{45}{8}\right)$.

Thus, the corner points of the feasible region are

$O(0, 0), C(6, 0), E\left(\frac{9}{4}, \frac{45}{8}\right)$ and $B\left(0, \frac{15}{2}\right)$.

Value of daily output $z = 60x + 40y$:

(i) At $O(0, 0)$ it is $z = (60 \times 0 + 40 \times 0) = 0$.

(ii) At $C(6, 0)$ it is $z = (60 \times 6 + 40 \times 0) = 360$.

(iii) At $E\left(\frac{9}{4}, \frac{45}{8}\right)$ it is $z = \left(60 \times \frac{9}{4} + 40 \times \frac{45}{8}\right) = 360$.

(iv) At $B\left(0, \frac{15}{2}\right)$ it is $z = \left(60 \times 0 + 40 \times \frac{15}{2}\right) = 300$.

Thus, either (6 machines of type A and no machine of type B) or (2 machines of type A and 6 machines of type B) be used to have maximum output.

[**NOTE** $\frac{9}{4}$ machines $\equiv$ 2 machines and $\frac{45}{8}$ machines $\equiv$ 6 machines.]

EXAMPLE 11 *A retired person has* ₹ 70000 *to invest and two types of bonds are available in the market for investment. First type of bond yields an annual income of* 8% *on the amount invested and the second type of bond yields* 10% *per annum. As per norms, he has to invest minimum of* ₹ 10000 *in the first type and not more than* ₹ 30000 *in the second type. How should he plan his investment, so as to get maximum return, after one year of investment?* **[CBSE 2007]**

SOLUTION Let bonds A be at 8% and bonds B be at 10%.

Suppose he plans to invest ₹ x in bonds A and ₹ y in bonds B.

Then, clearly $x + y = 70000$. ... (i)

He invests minimum of ₹ 10000 in bonds A.

$\therefore \quad x \geq 10000$. ... (ii)

Also, he invests not more than ₹ 30000 in bonds B.

$\therefore \quad y \leq 30000$ ... (iii)

Let z be the annual return on these investments. Then,

$$z = \frac{8x}{100} + \frac{10y}{100} \Rightarrow z = 0.08x + 0.1y. \quad \text{... (iv)}$$

Thus, we have to maximize z, subject to the conditions

$$\left.\begin{aligned} x + y &= 70000 \\ x &\geq 10000 \\ y &\leq 30000 \end{aligned}\right\}$$

Now, $x + y = 70000 \Rightarrow \frac{x}{70000} + \frac{y}{70000} = 1$.

This line meets the axes at $A(70000, 0)$ and $B(0, 70000)$.

Plot these points and join them to obtain the line $x + y = 70000$.

Clearly, $(0, 0)$ satisfies $x + y \leq 70000$.

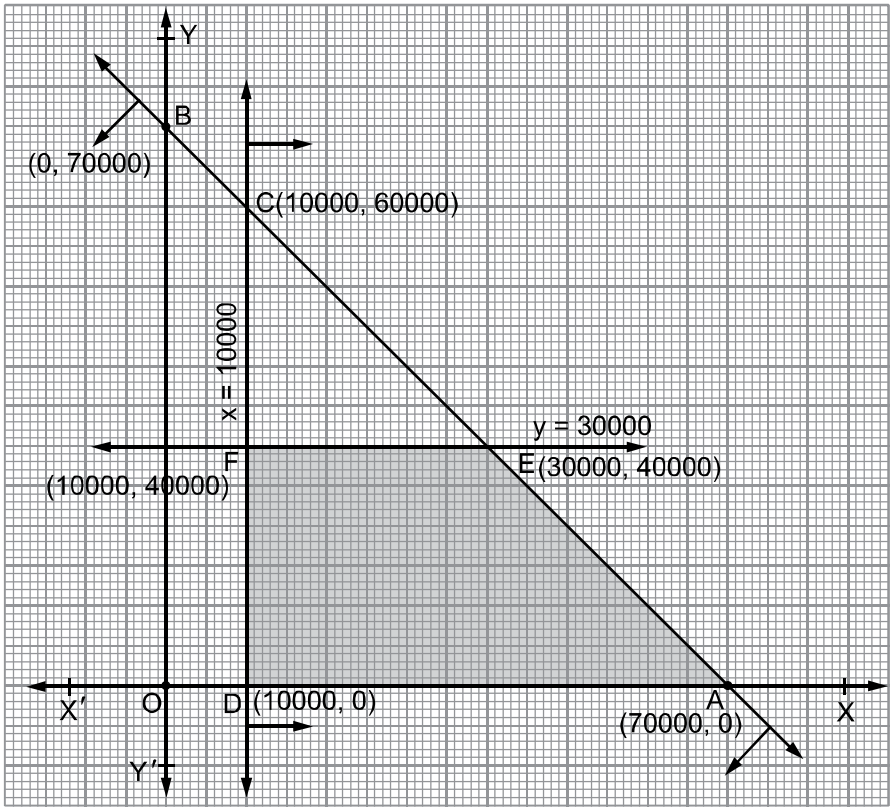
So, the region below AB represents $x + y \leq 70000$.

$x \geq 10000$ is the region parallel to the y-axis and to the right of it beyond the line $x = 10000$.

Thus, the region to the right of line CD represents $x \geq 10000$.

$y \leq 30000$ is the region parallel to the x-axis and above it, but below the line $y = 30000$.

Thus, the region below the line EF and above the x-axis represents $y \leq 30000$.



Thus, the corner points of the feasible region are

$D(10000, 0)$, $A(70000, 0)$, $E(30000, 40000)$ and $F(10000, 40000)$.

Value of annual return $z = 0.08x + 0.1y$:

(i) At $D(10000, 0)$ it is $z = (0.08 \times 10000 + 0.1 \times 0) = 800$.

(ii) At $A(70000, 0)$ it is $z = (0.08 \times 70000 + 0.1 \times 0) = 5600$.

(iii) At $E(30000, 40000)$ it is $z = (0.08 \times 30000 + 0.1 \times 40000) = 6400$.

(iv) At $F(10000, 40000)$ it is $z = (0.08 \times 10000 + 0.1 \times 40000) = 4800$.

So, in order to get a maximum annual return, he should invest ₹ 30000 in bond A and ₹ 40000 in bond B.

EXERCISE 33B

Long-Answer Questions

1. Find the maximum value of $Z = 7x + 7y$, subject to the constraints
$x \geq 0, y \geq 0, x + y \geq 2$ and $2x + 3y \leq 6$.

2. Maximize $Z = 4x + 9y$, subject to the constraints
$x \geq 0, y \geq 0, x + 5y \leq 200, 2x + 3y \leq 134$.

3. Find the maximum value of $Z = 3x + 5y$, subject to the constraints
$-2x + y \leq 4, x + y \geq 3, x - 2y \leq 2, x \geq 0$ and $y \geq 0$.

4. Minimize $Z = 2x + 3y$, subject to the constraints
$x \geq 0, y \geq 0, x + 2y \geq 1$ and $x + 2y \leq 10$.

5. Maximize $Z = 3x + 5y$, subject to the constraints
$x + 2y \leq 2000, x + y \leq 1500, y \leq 600, x \geq 0$ and $y \geq 0$.

6. Find the maximum and minimum values of $Z = 2x + y$, subject to the constraints
$$x + 3y \geq 6, x - 3y \leq 3, 3x + 4y \leq 24,$$
$$-3x + 2y \leq 6, 5x + y \geq 5, x \geq 0 \text{ and } y \geq 0.$$

7. Mr Dass wants to invest ₹ 12000 in Public Provident Fund (PPF) and in National bonds. He has to invest at least ₹ 1000 in PPF and at least ₹ 2000 in bonds. If the rate of interest on PPF is 12% per annum and that on bonds is 15% per annum, how should he invest the money to earn maximum annual income? Also find the maximum annual income.

8. A small firm manufactures necklaces and bracelets. The total number of necklaces and bracelets that it can handle per day is at most 24. It takes 1 hour to make a bracelet and half an hour to make a necklace. The maximum number of hours available per day is 16. If the profit on a necklace is ₹ 100 and that on a bracelet is ₹ 300, how many of each should be produced daily to maximize the profit? It is being given that at least one of each must be produced.

9. A man has ₹ 1500 to purchase rice and wheat. A bag of rice and a bag of wheat cost ₹ 180 and ₹ 120 respectively. He has a storage capacity of 10 bags only. He earns a profit of ₹ 11 and ₹ 8 per bag of rice and wheat respectively. How many bags of each must he buy to make maximum profit?

[CBSE 2008C, '09C]

10. A manufacturer produces nuts and bolts for industrial machinery. It takes 1 hour of work on machine A and 3 hours on machine B to produce a packet of nuts while it takes 3 hours on machine A and 1 hour on machine B to produce a packet of bolts. He earns a profit of ₹ 17.50 per packet on nuts and ₹ 7 per packet on bolts. How many packets of each should be produced each day so as to maximize his profit if he operates his machines for at the most 12 hours a day? Also find the maximum profit. **[CBSE 2009C, '12]**

11. Two tailors, A and B, earn ₹ 300 and ₹ 400 per day respectively. A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. How many days should each of them work if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labour cost?

12. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items. A fan costs him ₹ 360 and a sewing machine, ₹ 240. He expects to gain ₹ 22 on a fan and ₹ 18 on a sewing machine. Assuming that he can sell all the items he can buy, how should he invest the money in order to maximize the profit?

[CBSE 2009, '09C]

13. A firm manufactures two types of products, A and B, and sells them at a profit of ₹ 2 on type A and ₹ 2 on type B. Each product is processed on two machines, M_1 and M_2. Type A requires one minute of processing time on M_1 and two minutes on M_2. Type B requires one minute on M_1 and one minute on M_2. The machine M_1 is available for not more than 6 hours 40 minutes while M_2 is available for at most 10 hours a day.

    Find how many products of each type the firm should produce each day in order to get maximum profit.

14. A manufacturer produces two types of soap bars using two machines, A and B. A is operated for 2 minutes and B for 3 minutes to manufacture the first type, while it takes 3 minutes on machine A and 5 minutes on machine B to manufacture the second type. Each machine can be operated at the most for 8 hours per day. The two types of soap bars are sold at a profit of ₹ 0.25 and ₹ 0.50 each. Assuming that the manufacturer can sell all the soap bars he can manufacture, how many bars of soap of each type should be manufactured per day so as to maximize his profit?

15. A manufacturer of a line of patent medicines is preparing a production plan on medicines A and B. There are sufficient ingredients available to make 20000 bottles of A and 40000 bottles of B but there are only 45000 bottles into which either of the medicines can be put. Furthermore, it takes 3 hours to prepare enough material to fill 1000 bottles of A and it takes 1 hour to prepare enough material to fill 1000 bottles of B, and there are 66 hours available for this operation. The profit is ₹ 8 per bottle for A and ₹ 7 per bottle for B.

    How should the manufacturer schedule the production in order to maximize his profit? Also, find the maximum profit.

16. A toy company manufactures two types of dolls, A and B. Each doll of type B takes twice as long to produce as one of type A, and the company would have time to make a maximum of 2000 per day, if it produces only type A. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). Type B requires a fancy dress of which there are only 600 per day available. If the company makes profits of ₹ 3 and ₹ 5 per doll respectively on dolls A and B, how many of each should be produced per day in order to maximize the profit? Also, find the maximum profit.

17. A small manufacturer has employed 5 skilled men and 10 semiskilled men and makes an article in two qualities, a de luxe model and an ordinary model. The making of a de luxe model requires 2 hours' work by a skilled man and 2 hours' work by a semiskilled man. The ordinary model requires 1 hour by a skilled man and 3 hours by a semiskilled man. By union rules, no man can work more than 8 hours per day. The manufacturer gains ₹ 15 on the de luxe model and ₹ 10 on the ordinary model. How many of each type should be made in order to maximize his total daily profit? Also, find the maximum daily profit.

18. A company producing soft drinks has a contract which requires a minimum of 80 units of chemical A and 60 units of chemical B to go in each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier X has a mix of 4 units of A and 2 units of B that costs ₹ 10, and the supplier Y has a mix of 1 unit of A and 1 unit of B that costs ₹ 4. How many mixes from X and Y should the company purchase to honour the contract requirement and yet minimize the cost? **[CBSE 2012]**

19. A small firm manufactures gold rings and chains. The combined number of rings and chains manufactured per day is at most 24. It takes 1 hour to make a ring and half an hour for a chain. The maximum number of hours available per day is 16. If the profit on a ring is ₹ 300 and that on a chain is ₹ 190, how many of each should be manufactured daily so as to maximize the profit?

20. A manufacturer makes two types, A and B, of teapots. Three machines are needed for the manufacture and the time required for each teapot on the machines is given below.

Machine / Type	Time (in minutes)		
	I	II	III
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each teapot of type A is 75 paise and that on each teapot of type B is 50 paise, show that 15 teapots of type A and 30 of type B should be manufactured in a day to get the maximum profit.

21. A manufacturer makes two products, A and B. Product A sells at ₹ 200 each and takes $\frac{1}{2}$ hour to make. Product B sells at ₹ 300 each and takes 1 hour to make. There is a permanent order for 14 of product A and 16 of product B. A working week consists of 40 hours of production and the weekly turnover must not be less than ₹ 10000. If the profit on each of the product A is ₹ 20 and on product B, it is ₹ 30 then how many of each should be produced so that the profit is maximum? Also, find the maximum profit.

22. A man owns a field of area 1000 m^2. He wants to plant fruit trees in it. He has a sum of ₹ 1400 to purchase young trees. He has the choice of two types of trees. Type *A* requires 10 m^2 of ground per tree and costs ₹ 20 per tree, and type *B* requires 20 m^2 of ground per tree and costs ₹ 25 per tree. When full grown, a type-*A* tree produces an average of 20 kg of fruit which can be sold at a profit of ₹ 2 per kg and a type-*B* tree produces an average of 40 kg of fruit which can be sold at a profit of ₹ 1.50 per kg. How many of each type should be planted to achieve maximum profit when trees are fully grown? What is the maximum profit?

23. A publisher sells a hardcover edition of a book for ₹ 72 and a paperback edition of the same for ₹ 40. Costs to the publisher are ₹ 56 and ₹ 28 respectively in addition to weekly costs of ₹ 9600. Both types require 5 minutes of printing time although the hardcover edition requires 10 minutes of binding time and the paperback edition requires only 2 minutes. Both the printing and binding operations have 4800 minutes available each week. How many of each type of books should be produced in order to maximize the profit? Also, find the maximum profit per week.

24. A gardener has a supply of fertilizers of the type I which consists of 10% nitrogen and 6% phosphoric acid, and of the type II which consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, he finds that he needs at least 14 kg of nitrogen and 14 kg of phosphoric acid for his crop. If the type-I fertilizer costs 60 paise per kg and the type-II fertilizer costs 40 paise per kg, determine how many kilograms of each type of fertilizer should be used so that the nutrient requirements are met at a minimum cost. What is the minimum cost? **[CBSE 2008]**

25. Two godowns, *A* and *B*, have a grain storage capacity of 100 quintals and 50 quintals respectively. Their supply goes to three ration shops, *D*, *E* and *F*, whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table.

	Cost of transportation (in ₹ per quintal)	
From / To	*A*	*B*
D	6.00	4.00
E	3.00	2.00
F	2.50	3.00

How should the supplies be transported in order that the transportation cost is minimum?

26. A brick manufacturer has two depots, P and Q, with stocks of 30000 and 20000 bricks respectively. He receives orders from three buildings A, B, C, for 15000, 20000 and 15000 bricks respectively. The cost of transporting 1000 bricks to the buildings from the depots are given below.

	Cost of transportation (in ₹ per 1000 bricks)		
From \ To	A	B	C
P	40	20	30
Q	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum?

27. A medicine company has factories at two places, X and Y. From these places, supply is made to each of its three agencies situated at P, Q and R. The monthly requirements of the agencies are respectively 40 packets, 40 packets and 50 packets of medicines, while the production capacity of the factories at X and Y are 60 packets and 70 packets respectively. The transportation cost per packet from the factories to the agencies are given as follows.

	Transportation cost per packet (in ₹)	
To \ From	X	Y
P	5	4
Q	4	2
R	3	5

How many packets from each factory should be transported to each agency so that the cost of transportation is minimum? Also, find the minimum cost.

28. An oil company has two depots, A and B, with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E, F, whose requirements are 4500 L, 3000 L and 3500 L respectively. The distances (in km) between the depots and the petrol pumps are given in the following table:

	Distance (in km)	
From / To	*A*	*B*
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost per km is Re 1 per litre, how should the delivery be scheduled in order that the transportation cost is minimum?

29. A firm is engaged in breeding pigs. The pigs are fed on various products grown on the farm. They need certain nutrients, named as *X*, *Y*, *Z*. The pigs are fed on two products, *A* and *B*. One unit of product *A* contains 36 units of *X*, 3 units of *Y* and 20 units of *Z*, while one unit of product *B* contains 6 units of *X*, 12 units of *Y* and 10 units of *Z*. The minimum requirements of *X*, *Y*, *Z* are 108 units, 36 units and 100 units respectively. Product *A* costs ₹ 20 per unit and product *B* costs ₹ 40 per unit. How many units of each product must be taken to minimize the cost? Also, find the minimum cost.

30. A dietician wishes to mix two types of food, *X* and *Y*, in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food *X* contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C, while food *Y* contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5 per kg to purchase the food *X* and ₹ 7 per kg to purchase the food *Y*. Determine the minimum cost of such a mixture.

[CBSE 2011C, '12]

31. A diet for a sick person must contain at least 4000 units of vitamins, 50 units of minerals and 1400 calories. Two foods, *A* and *B*, are available at a cost of ₹ 4 and ₹ 3 per unit respectively. If one unit of *A* contains 200 units of vitamins, 1 unit of minerals and 40 calories, and 1 unit of *B* contains 100 units of vitamins, 2 units of minerals and 40 calories, find what combination of foods should be used to have the least cost.

[CBSE 2008]

32. A housewife wishes to mix together two kinds of food, *X* and *Y*, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. **[CBSE 2009C]**

The vitamin contents of 1 kg of each food are given below.

	Vitamin A	Vitamin B	Vitamin C
Food *X*	1	2	3
Food *Y*	2	2	1

If 1 kg of food *X* costs ₹ 6 and 1 kg of food *Y* costs ₹ 10, find the minimum cost of the mixture which will produce the diet.

33. A firm manufactures two types of products, A and B, and sells them at a profit of ₹ 5 per unit of type A and ₹ 3 per unit of type B. Each product is processed on two machines, M_1 and M_2. One unit of type A requires one minute of processing time on M_1 and two minutes of processing time on M_2; whereas one unit of type B requires one minute of processing time on M_1 and one minute on M_2. Machines M_1 and M_2 are respectively available for at most 5 hours and 6 hours in a day. Find out how many units of each type of product the firm should produce a day in order to maximize the profit. Solve the problem graphically. **[CBSE 2000]**

34. A small firm manufactures items A and B. The total number of items that it can manufacture in a day is at the most 24. Item A takes one hour to make while item B takes only half an hour. The maximum time available per day is 16 hours. If the profit on one unit of item A be ₹ 300 and that on one unit of item B be ₹ 160, how many of each type of item should be produced to maximize the profit? Solve the problem graphically. **[CBSE 2000, '04]**

35. A manufacturer produces two types of steel trunks. He has two machines, A and B. The first type of trunk requires 3 hours on machine A and 3 hours on machine B. The second type requires 3 hours on machine A and 2 hours on machine B. Machines A and B can work at most for 18 hours and 15 hours per day respectively. He earns a profit of ₹ 30 and ₹ 25 per trunk of the first type and second type respectively. How many trunks of each type must he make each day to make the maximum profit? **[CBSE 2005, '12]**

36. A company manufactures two types of toys, A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. The profit is ₹ 50 each on type A and ₹ 60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit? **[CBSE 2001]**

37. Kellogg is a new cereal formed of a mixture of bran and rice, that contains at least 88 grams of protein and at least 36 milligrams of iron. Knowing that bran contains 80 grams of protein and 40 milligrams of iron per kilogram, and that rice contains 100 grams of protein and 30 milligrams of iron per kilogram, find the minimum cost of producing this new cereal if bran costs ₹ 5 per kilogram and rice costs ₹ 4 per kilogram. **[CBSE 2002]**

38. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items. A fan costs him ₹ 360 and a sewing machine ₹ 240. He expects to sell a fan at a profit of ₹ 22 and a sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he buys, how should he invest his money to maximize the profit? Solve graphically and find the maximum profit. **[CBSE 2002, '03, '06]**

39. Anil wants to invest at the most ₹ 12000 in bonds A and B. According to rules, he has to invest at least ₹ 2000 in Bond A and at least ₹ 4000 in Bond B. If the rate of interest of Bond A is 8% per annum and on Bond B, it is 10% per annum, how should he invest his money for maximum interest? Solve the problem graphically. **[CBSE 2004C]**

40. Maximize $z = 60x + 15y$, subject to the constraints
$x + y \leq 50,\ \ 3x + y \leq 90,\ \ x, y \geq 0.$ **[CBSE 2005]**

41. A company manufactures two types of toys A and B. Type A requires 5 minutes each for cutting and 10 minutes each for assembling. Type B requires 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours available for cutting and 4 hours available for assembling in a day. He earns a profit of ₹ 50 each on type A and ₹ 60 each on type B. How many toys of each type should the company manufacture in a day to maximize the profit? **[CBSE 2007]**

42. One kind of cake requires 200 g of flour and 25 g of fat, another kind of cake requires 100 g of flour and 50 g of fat. Find the maximum number of cakes which can be made from 5 kg of flour and 1 kg of fat, assuming that there is no shortage of the other ingredients used in making the cakes. Make it an LPP and solve it graphically. **[CBSE 2014C]**

43. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹ 120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week? **[CBSE 2014]**

ANSWERS (EXERCISE 33B)

1. Maximum $Z = 21$ at $x = 3,\ y = 0$

2. Maximum $Z = 382$ when $x = 10$ and $y = 38$

3. Minimum $Z = 9\frac{2}{3}$ at $\left(\frac{8}{3}, \frac{1}{3}\right)$ **4.** Minimum $Z = \frac{3}{2}$ at $\left(0, \frac{1}{2}\right)$

5. Maximum $Z = 5500$ at $x = 1000$ and $y = 500$

6. Maximum $Z = 14\frac{1}{3}$ at $\left(\frac{84}{13}, \frac{15}{13}\right)$ and minimum $Z = 3\frac{1}{14}$ at $\left(\frac{9}{14}, \frac{25}{14}\right)$

7. Rs 1000 in PPF and ₹ 11000 in bonds; ₹ 1770

8. $x = 16$ and $y = 8$ **9.** 5 bags of each

10. Three packets each of nuts and bolts; maximum profit = ₹ 73.50

11. 5 days and 3 days

12. 24 sewing machines only **13.** 200, 200

14. For maximum profit, 96 soap bars of 2nd type must be manufactured.

15. 10500 bottles of A and 34500 bottles of B, and maximum profit is ₹ 325500

16. 1000 of type A and 500 of type B, maximum profit = ₹ 5500

17. 10 de luxe, 20 ordinary; maximum daily gain = ₹ 400

18. 10 mixes from X and 40 mixes from Y

19. Number of rings = 8, number of chains = 16

21. $A = 48$, $B = 16$; maximum profit = ₹ 1440

22. Type $A = 20$, type $B = 40$; maximum profit = ₹ 3200

23. 360 and 600; ₹ 2880

24. 100 kg of type I and 80 kg of type II; minimum cost = ₹ 92

25. 10 q, 50 q and 40 q from A to D, E, F respectively, and 50 q, 0 q and 0 q from B to D, E, F respectively

26. From P: 0, 20000, 10000 bricks to A, B, C respectively
From Q: 15000, 0, 5000 bricks to A, B, C respectively

27. From X: 10 pkt, 0 pkt and 50 pkt to P, Q, R respectively
From Y: 30 pkt, 40 pkt and 0 pkt to P, Q, R respectively
Minimum cost = ₹ 400

28. From A: 500 L, 3000 L, 3500 L to D, E, F respectively
From B: 4000 L, 0 L, 0 L to D, E, F respectively

29. 2 units of A, 4 units of B; minimum cost = ₹ 160

30. $x = 2$, $y = 4$; ₹ 38 **31.** 5 units of A and 30 units of B **32.** ₹ 52

33. Z is maximum at $P(60, 240)$ and its maximum value is ₹ 1020.

34. Z is maximum at (8, 16) and its maximum value is ₹ 4960.

35. For getting a maximum profit of ₹ 165, 3 trunks of each type should be manufactured.

36. For getting a maximum profit of ₹ 1500, 12 toys of type A and 15 toys of type B should be manufactured.

37. The minimum cost of producing this cereal is ₹ 4.60 per kg.

38. The profit is maximum at $E(8, 12)$ and it is ₹ 392.

39. The interest is maximum at $E(2000, 10000)$ and it is ₹ 1160.

40. The maximum value of z is at $C(30, 0)$.

41. 12 toys of type A and 15 toys of type B to earn maximum profit ₹ 1500

42. Max. no. of cakes: 1st kind—20, 2nd kind—10

43. For max. profit, 12 pieces of type A and 8 pieces of type B; max. profit = ₹ 1680

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 33B)

7. Maximize $Z = \frac{12x}{100} + \frac{15y}{100}$ subject to $x \geq 1000, y \geq 2000, x + y \leq 12000.$

8. Maximize $P = 100x + 300y$ subject to $x > 0, y > 0, x + y \leq 24, \frac{1}{2}x + y \leq 16.$

10. Maximize $P = 17.5x + 7y$ subject to $x \geq 0, y \geq 0, x + 3y \leq 12$ and $3x + y \leq 12.$

11. Let A and B work for x and y days respectively.
Minimize $Z = 300x + 400y$ subject to $x \geq 0, y \geq 0, 6x + 10y \geq 60$ and $4x + 4y \geq 32.$

12. Maximize $P = 22x + 18y$ subject to
$x \geq 0, y \geq 0, x + y \leq 20, 360x + 240y \leq 5760.$

13. Maximize $Z = 2x + 2y$ subject to $x \geq 0, y \geq 0, x + y \leq 400, 2x + y \leq 600.$

14. Maximize $P = \frac{x}{4} + \frac{y}{2}$ subject to $x \geq 0, y \geq 0, 2x + 3y \leq 480$ and $3x + 5y \leq 480.$

15. Maximize $Z = 8x + 7y$ subject to $3x + y \leq 66000, x + y \leq 45000, x \leq 20000, y \leq 40000,$ $x \geq 0$ and $y \geq 0.$

16. Maximize $Z = 3x + 5y$ subject to $x + 2y \leq 2000, x + y \leq 1500, y \leq 600, x \geq 0$ and $y \geq 0.$

17. Suppose x de luxe models and y ordinary models of articles be made each day. Then,
maximize $Z = 15x + 10y$, subject to the constraints
$x \geq 0, y \geq 0, 2x + y \leq 40$ and $2x + 3y \leq 80.$

18. Let the company purchase x mixes from X and y mixes from Y. Then,
minimize $Z = 10x + 4y$, subject to the constraints
$x \geq 0, y \geq 0, 4x + y \geq 80$ and $2x + y \geq 60.$

20. Let x teapots of type A and y teapots of type B be manufactured. Then,
$x \geq 0, y \geq 0, 12x + 6y \leq 6 \times 60, 18x + 0y \leq 6 \times 60, 6x + 9y \leq 6 \times 60$
$\Rightarrow \quad x \geq 0, y \geq 0, 2x + y \leq 60, x \leq 20, 2x + 3y \leq 120.$
Profit function is $Z = \frac{75}{100}x + \frac{50}{100}y \quad \Rightarrow \quad Z = \frac{3}{4}x + \frac{1}{2}y.$
Maximize Z, subject to the constraints
$x \geq 0, y \geq 0, x \leq 20, 2x + y \leq 60, 2x + 3y \leq 120.$
Then $x = 15$ and $y = 30$ will give the maximum value of Z.

21. Let the number of articles produced per week be x of A and y of B.
Then, $\frac{1}{2}x + y \leq 40, 200x + 300y \geq 10000, x \geq 14, y \geq 16.$
Profit function is $Z = 20x + 30y.$
Maximize $Z = 20x + 30y$, subject to the constraints
$x + 2y \leq 80, 2x + 3y \geq 1000, x \geq 14, y \geq 16.$

22. Let x plants of type A and y plants of type B be planted.
Then, $x \geq 0, y \geq 0, 20x + 25y \leq 1400, 10x + 20y \leq 1000.$
Profit function is $Z = 2 \times 20x + \frac{3}{2} \times 40y$, i.e., $Z = 40x + 60y.$

23. Let x copies of the hardcover edition and y copies of the paperback edition be prepared. Then,

$C = 9600 + 56x + 28y$ and $S = 72x + 40y$.

Profit function is $Z = (S - C)$, i.e., $Z = 16x + 12y - 9600$.

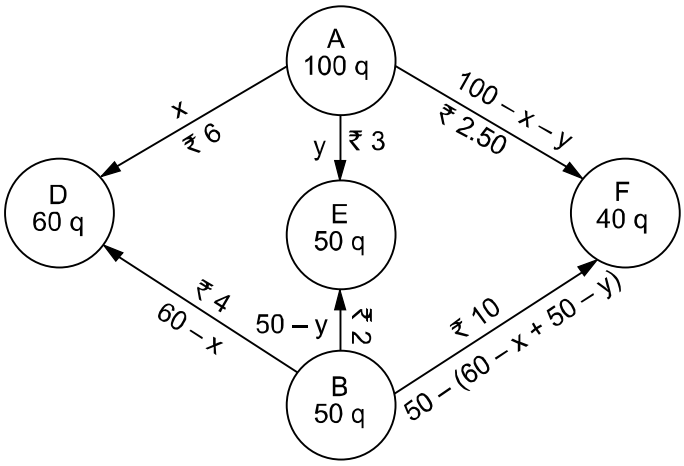
Maximize Z subject to $x \geq 0, y \geq 0, (5x + 10x) + (5y + 2y) \leq 4800$.

24. Let x kg of type I and y kg of type II be used. Then,

$$x \geq 0, y \geq 0, \frac{10}{100}x + \frac{5}{100}y \geq 14, \frac{6}{100}x + \frac{10}{100}y \geq 14.$$

Cost function is $Z = \frac{60}{100}x + \frac{40}{100}y$, i.e., $Z = \frac{3}{5}x + \frac{2}{5}y$.

25. The entire problem can be explained diagramatically as shown below.



Let x q and y q of grains be transported from A to shops D and E respectively. Then, $[100 - (x + y)]$ q will be transported from A to F.

$\therefore \quad x \geq 0, y \geq 0, 100 - (x + y) \geq 0 \Rightarrow x + y \leq 100.$

D, E, F require 60 q, 50 q and 40 q respectively.

From B, there will be a supply of $(60 - x)$q to D; $(50 - y)$q to E and $[50 - (60 - x + 50 - y)]$ q to F.

$\therefore \quad (60 - x) \geq 0, (50 - y) \geq 0, (x + y - 60) \geq 0$

$\Rightarrow \quad x \leq 60, y \leq 50$ and $x + y \geq 60.$

$\therefore$ cost of transportation is given by

$$Z = 6x + 4(60 - x) + 3y + 2(50 - y) + \frac{5}{2} + (100 - x - y) + 3(x + y - 60)$$

$$\Rightarrow \quad Z = \frac{5}{2}x + \frac{3}{2}y + 410.$$

28. Let x and y litres be transported from A to D and E.

Then, $(7000 - x - y)$ litres will be transported to F from A.

From B, $(4500 - x)$ litres, $(3000 - y)$ litres, $[4000 - (4500 - x + 3000 - y)]$ litres will be transported to D, E and F respectively.

Cost of transportation is

$$Z = 7x + 6y + 3(7000 - x - y) + 3(45000 - x) + 4(3000 - y) + 2(x + y - 3500).$$

Minimize $Z = 3x + y + 39500$, subject to

$x \geq 0, y \geq 0, x + y \leq 7000, x \leq 4500, y \leq 3000, x + y \geq 3500$ and $x = 500$.

29. Let x units of A and y units of B be taken. Then,
minimize $C = 20x + 40y$, subject to
$x \geq 0, y \geq 0,\ 36x + 6y \geq 108,\ 3x + 12y \geq 36,\ 20x + 10y \geq 100.$

30. Let x kg of the food X and y kg of the food Y be taken. Then,
$x \geq 0, y \geq 0, 2x + y \geq 8$ and $x + 2y \geq 10.$
The cost function is $Z = 5x + 7y.$

31. Let x units of the food A and y units of the food B be taken. Then,
$x \geq 0, y \geq 0, 200x + 100y \geq 4000, x + 2y \geq 50, 40x + 40y \geq 1400.$
The cost function is $Z = 4x + 3y.$

32. Let x kg of the food X and y kg of the food Y be taken. Then,
$x \geq 0, y \geq 0, x + 2y \geq 10, 2x + 2y \geq 12$ and $3x + y \geq 8.$
The cost function is $Z = 6x + 10y.$

33. Let x units of the type A and y units of the type B be produced.
Then, $x_1 + x_2 \leq 300; 2x_1 + x_2 \leq 360; x_1 \geq 0$ and $x_2 \geq 0.$
Maximize $Z = 5x + 3y.$

34. Let x items of A and y items of B be produced. Then,
$$x + y \leq 24, 1 \cdot x + \frac{1}{2} \cdot y \leq 16; x \geq 0 \text{ and } y \geq 0.$$
Maximize $P = 300x + 160y.$

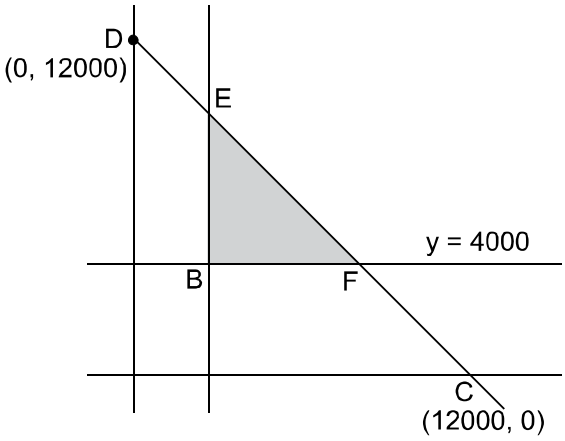
35. Let x trunks of the first type and y trunks of the second type be manufactured. Then,
$3x + 3y \leq 18;\ 3x + 2y \leq 15;\ x \geq 0$ and $y \geq 0.$
Maximize $Z = 30x + 25y.$

36. Let x and y be the number of toys of type A and type B respectively. Then,
$5x + 8y \leq 180,\ 10x + 8y \leq 240$ and $x \geq 0,\ y \geq 0.$
Maximize $Z = 50x + 60y.$

37. Let the cereal contain x kg of bran and y kg of rice.
Maximize $Z = 5x + 4y$, subject to the conditions
$$\left(x \times \frac{80}{1000}\right) + \left(y \times \frac{100}{1000}\right) \geq \frac{88}{1000};$$
$$\left(x \times \frac{40}{1000}\right) + \left(y \times \frac{30}{1000}\right) \geq \frac{36}{1000};$$
$x \geq 0$ and $y \geq 0.$

38. Let the number of fans and sewing machines bought be x and y respectively.
Maximize $Z = 22x + 18y$, subject to
$x + y \leq 20,\ 360x + 240y \leq 5760$ and $x \geq 0, y \geq 0.$

39. Maximize $Z = \frac{8x}{100} + \frac{10y}{100} = \left(\frac{2x}{25} + \frac{y}{10}\right)$ subject to $x \geq 2000, y \geq 4000$ and $x + y \leq 12000.$
Draw the graphs of $x = 2000,\ y = 4000$ and $\frac{x}{12000} + \frac{y}{12000} = 1.$
Shade $x \geq 2000,\ y \geq 4000$ and $\frac{x}{12000} + \frac{y}{12000} \leq 1$ to give $\triangle BEF.$



Values of $\left(\frac{2x}{25}+\frac{y}{10}\right)$ at B, E and F are respectively 560, 1160 and 1040.

So, it is maximum at $E(2000, 10000)$.

41. Maximize $z = 50x + 60y$, subject to the constraints

$5x + 8y \le 180$,

$5x + 4y \le 120$,

$x \ge 0$ and $y \ge 0$.

The corner points of the feasible region are $O(0, 0), C(24, 0), D(0, 22.5)$ and $E(12, 15)$.

z is maximum at $E(12, 15)$.

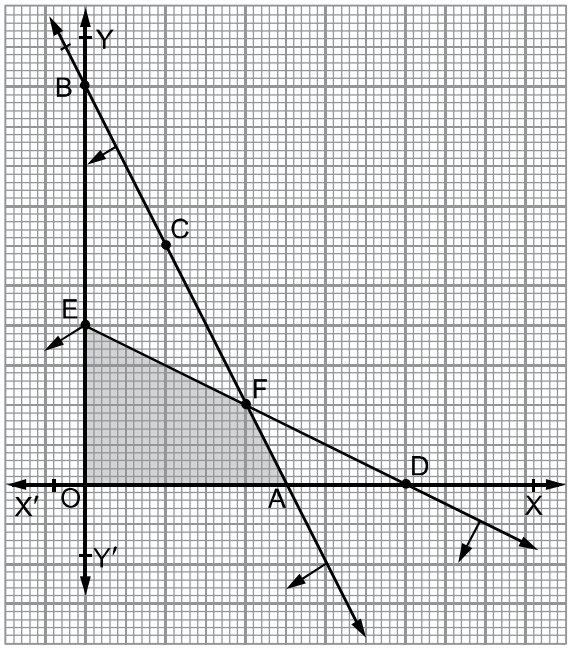
42. Let the number of cakes of the first and second type be x and y respectively.

Then, we have to maximise $z = x + y$ subject to the constraints

$200x + 100y \le 5000 \Rightarrow 2x + y \le 50$ ... (i)

$25x + 50y \le 1000 \Rightarrow x + 2y \le 40$ ... (ii)

and $x \ge 0, y \ge 0$.



First we draw the line $2x + y = 50$.

For this, we plot the points $A(25, 0)$, $B(0, 50)$ and $C(10, 30)$.

Draw the line ACB.

The region below this line represents $2x + y \leq 50$.

Now, we draw the line $x + 2y = 40$.

For this, we plot the points $D(40, 0)$, $E(0, 20)$ and $F(20, 10)$.

Draw the line DFE.

The region below this line represetns $x + 2y \leq 40$.

The feasible region contains the points $A(25, 0), F(20, 10)$ and $E(0, 20)$.

Value of z at $A(25, 0) = 25 + 0 = 25$.

Value of z at $F(20, 10) = 20 + 10 = 30$.

Value of z at $E(0, 20) = 0 + 20 = 20$.

For maximum value of z, we have $x = 20$ and $y = 10$.

Maximum number of cakes required for first and second type are 20 and 10 respectively.

43. Let the number of pieces of type A and type B manufactured per week be x and y respectively.

Then, we have to maximize $P = 80x + 120y$ subject to the constraints

$$9x + 12y \leq 180 \Rightarrow 3x + 4y \leq 60, \quad \text{... (i)}$$

$$x + 3y \leq 30 \quad \text{... (ii)}$$

and $x \geq 0, y \geq 0$.

We leave it to the reader to draw the graphs.

For maximum profit, we shall have 12 pieces of type A and 6 pieces of type B; and the maximum profit is ₹ 1680.